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A Note on the Extreme Points of Positive Quantum Operations

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Abstract In this paper, an error in the proof of Theorem 4.9 in Gudder's paper (Int. J. Theor. Phys. 47(1):268–279, 2008) is pointed out and it is proved that if $\mathcal{E} \approx \{E_1, E_2, \dots, E_n\} \in \mathcal{Q}_{pos}(\mathcal{H}) \setminus \mathcal{Q}_{pro}(\mathcal{H})$ such that $E_i \in \mathbb{C}I \setminus \{0\}$ and $E_j \notin \mathbb{C}I$ for some i, j in $\{1, 2, \dots, n\}$, then $\mathcal{E} \notin \mathrm{Ext}[\mathcal{Q}_{pos}(\mathcal{H})]$.

Keywords Quantum operation · Extreme point

1 Introduction

Recently, Gudder [1, 2] have established mathematical theory of the duality computer proposed by Long in [3, 4].

Let $\mathcal{H}$ be a separable complex Hilbert space and let $B(\mathcal{H})$ denote the all of bounded linear operators on $\mathcal{H}$. An operator $A \in B(\mathcal{H})$ is said to be a positive operator if $\langle Ax, x\rangle \geq 0$ for $x \in H$. An operator $P \in B(\mathcal{H})$ is said to be an orthogonal projection if $P^* = P = P^2$, where P^* denotes the adjoint of P. Now operators E that satisfy $0 \leq E \leq I$ are called effects. We denote the convex set of effects on $\mathcal{H}$ by $\mathscr{E}(\mathcal{H})$ and the set of projection operators on $\mathcal{H}$ by $\mathcal{P}(\mathcal{H})$.

Recall that a quantum operation on $\mathcal{H}$ is an endomorphism $\mathcal{E} : B(\mathcal{H}) \to B(\mathcal{H})$ of the form

$$\mathcal{E}(A) = \sum_{i=1}^{n} E_i A E_i^*$$

where $E_i \in B(\mathcal{H})$ $(1 \leq i \leq n)$ satisfy $\sum_{i=1}^{n} E_i^* E_i = I$ (see [1]). The elements of $\{E_1, E_2, \dots, E_n\}$ are called operational elements of $\mathcal{E}$ and denoted as $\mathcal{E} \approx \{E_1, E_2, \dots, E_n\}$. We

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say that a quantum operation $\mathcal{E} \approx \{E_1, E_2, \ldots, E_n\}$ is normal, positive, or projective if all E_i are normal, positive, or projection operators, respectively. If $E_i = \sqrt{p_i} U_i$ where U_i are unitary operators and p_i's are nonnegative real numbers such that $\sum_{i=1}^n p_i = 1$, then $\mathcal{E} \approx \{E_1, E_2, \ldots, E_n\}$ is called a unitary quantum operation. Let $\mathcal{Q}(\mathcal{H})$ denote the set of quantum operations on $\mathcal{H}$ and $\mathcal{Q}_u(\mathcal{H})$, $\mathcal{Q}_{pos}(\mathcal{H})$, and $\mathcal{Q}_{pro}(\mathcal{H})$ be the sets of all unitary, positive and projective quantum operations on $\mathcal{H}$, respectively. It is easy to check that $\mathcal{Q}(\mathcal{H})$, $\mathcal{Q}_u(\mathcal{H})$ and $\mathcal{Q}_{pos}(\mathcal{H})$ are convex sets, while $\mathcal{Q}_{pro}(\mathcal{H})$ is not convex. In papers [1, 2], it was shown that $\mathcal{Q}_{pro}(\mathcal{H}) \subset \mathcal{Q}_u(\mathcal{H})$, $\mathcal{P}(\mathcal{H}) = \mathrm{Ext}[\mathscr{E}(\mathcal{H})]$, $\mathcal{Q}_{pro}(\mathcal{H}) \subset \mathrm{Ext}[\mathcal{Q}_{pos}(\mathcal{H})]$, the elements of $\mathrm{Ext}[\mathcal{Q}_u(H)]$ are precisely those of the form $\mathcal{E}(A) = UAU^*$ where U is a unitary operator on $\mathcal{H}$. Gudder in [2] proved that If $\dim \mathcal{H} < \infty$, then for every $\mathcal{E} \in \mathcal{Q}_{pos}(\mathcal{H})$ is a convex combination of elements of $\mathcal{Q}_{pro}(\mathcal{H})$. Also, Gudder in [2] pointed a result (Theorem 4.9) which says that

Theorem (Gudder [2, Theorem 4.9]) *If $\mathcal{E} \approx \{E_1, E_2, \ldots, E_n\} \in \mathcal{Q}_{pos}(\mathcal{H}) \setminus \mathcal{Q}_{pro}(\mathcal{H})$ and at least one of the E_i is invertible, then $\mathcal{E} \notin Ext[\mathcal{Q}_{pos}(\mathcal{H})]$.*

This result seems to be interesting and correct. In the proof given in [2] does not work well because that the following inequality

$$[(1+\varepsilon)ab - \varepsilon]^2 \le [(1+\varepsilon)a^2 - \varepsilon][(1+\varepsilon)b^2 - \varepsilon] \tag{1}$$

was used [2, p. 278, 17], which is not valid unless $a = b$! Clearly, the inequality (1) is the same as $2ab \ge a^2 + b^2$, that is $a = b$ since $2ab \le a^2 + b^2$ is always valid. This implies that the Gudder's proof is incorrect. However, we can neither give a revised proof nor find a counterexample to show that it is not correct.

In this note, we will give a revised of Gudder's result above and prove that if $\mathcal{E} \approx \{E_1, E_2, \ldots, E_n\} \in \mathcal{Q}_{pos}(\mathcal{H})$ such that $E_i \in \mathbb{C}I \setminus \{0\}$ and $E_j \notin \mathbb{C}I$ for some i, j in $\{1, 2, \ldots, n\}$, then $\mathcal{E} \notin \mathrm{Ext}[\mathcal{Q}_{pos}(\mathcal{H})]$.

2 Main Results

Theorem 1 *If $\mathcal{E} \approx \{E_1, E_2, \ldots, E_n\} \in \mathcal{Q}_{pos}(\mathcal{H}) \setminus \mathcal{Q}_{pro}(\mathcal{H})$ such that $E_i \in \mathbb{C}I \setminus \{0\}$ and $E_j \notin \mathbb{C}I$ for some i, j in $\{1, 2, \ldots, n\}$, then $\mathcal{E} \notin Ext[\mathcal{Q}_{pos}(\mathcal{H})]$.*

Proof We may assume that $E_1 = \lambda I$ where $\lambda > 0$ and $E_2 \notin \mathbb{C}I$. Take an ε such that $0 < \varepsilon < (1+\varepsilon)\lambda^2$. Clearly,

$$\begin{aligned}
\mathcal{E}(A) &= \sum_{i=1}^n E_i A E_i^* \\
&= \frac{1}{2}\left[\sum_{i=1}^n \sqrt{1-\varepsilon} E_i A \sqrt{1-\varepsilon} E_i^* + \sqrt{\varepsilon} I A \sqrt{\varepsilon} I\right] \\
&\quad + \frac{1}{2}\left[\sum_{i=1}^n \sqrt{1+\varepsilon} E_i A \sqrt{1+\varepsilon} E_i^* - \sqrt{\varepsilon} I A \sqrt{\varepsilon} I\right] \\
&= \frac{1}{2}\mathcal{E}_1(A) + \frac{1}{2}\mathcal{E}_2(A),
\end{aligned}$$

for all $A \in B(\mathcal{H})$ where

$$\mathcal{E}_1(A) = \sum_{i=1}^{n} \sqrt{1-\varepsilon} E_i A \sqrt{1-\varepsilon} E_i^* + \sqrt{\varepsilon} I A \sqrt{\varepsilon} I,$$

$$\mathcal{E}_2(A) = \sum_{i=1}^{n} \sqrt{1+\varepsilon} E_i A \sqrt{1+\varepsilon} E_i^* - \sqrt{\varepsilon} I A \sqrt{\varepsilon} I.$$

It is easy to check that $\mathcal{E}_1 \in \mathcal{Q}_{pos}(\mathcal{H})$. To show that $\mathcal{E}_2 \in \mathcal{Q}_{pos}(\mathcal{H})$, we put

$$B = \sqrt{(1+\varepsilon)\lambda^2 - \varepsilon} I.$$

Then

$$\mathcal{E}_2(A) = \sum_{i=2}^{n} \sqrt{1+\varepsilon} E_i A \sqrt{1+\varepsilon} E_i^* + BAB^*, \quad \forall A \in B(\mathcal{H}).$$

Since

$$\begin{aligned}
&\sum_{i=2}^{n} (\sqrt{1+\varepsilon} E_i)^2 + B^2 \\
&\quad = (1+\varepsilon) \sum_{i=1}^{n} E_i^2 - (1+\varepsilon) E_1^2 + [(1+\varepsilon)\lambda^2 - \varepsilon] I \\
&\quad = (1+\varepsilon) I - (1+\varepsilon)\lambda^2 I + [(1+\varepsilon)\lambda^2 - \varepsilon] I \\
&\quad = I,
\end{aligned}$$

we know that $\mathcal{E}_2 \in \mathcal{Q}_{pos}(\mathcal{H})$. Suppose that $\mathcal{E}_1(A) = \mathcal{E}_2(A)$ for all $A \in B(\mathcal{H})$. Then

$$\sum_{i=1}^{n} E_i A E_i^* = A, \quad \forall A \in B(\mathcal{H}).$$

It follows from [5, Lemma 3.2] that $E_i \in \mathbb{C}I$ for all $i \in \{1, 2 \ldots, n\}$. This contradicts the assumption that E_2 is not in $\mathbb{C}I$. Therefore, $\mathcal{E} \notin \mathrm{Ext}[\mathcal{Q}_{pos}(\mathcal{H})]$. The proof is completed. □

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